

# Information entropy and nucleon correlations in nuclei

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## Abstract

The information entropies in coordinate and momentum spaces and their sum ( $S_r$ ,  $S_k$ ,  $S$ ) are evaluated for many nuclei using "experimental" densities or/and momentum distributions. The results are compared with the harmonic oscillator model and with the short-range correlated distributions. It is found that  $S_r$  depends strongly on  $\ln A$  and does not depend very much on the model. The behaviour of  $S_k$  is opposite. The various cases we consider can be classified according to either the quantity of the experimental data we use or by the values of  $S$ , i.e., the increase of the quality of the density and of the momentum distributions leads to an increase of the values of  $S$ . In all cases, apart from the linear relation  $S = a + b \ln A$ , the linear relation  $S = a_V + b_V \ln V$  also holds.  $V$  is the mean volume of the nucleus. If  $S$  is considered as an ensemble entropy, a relation between  $A$  or  $V$  and the ensemble volume can be found. Finally, comparing different electron scattering experiments for the same nucleus, it is found that the larger the momentum transfer ranges, the larger the information entropy is. It is concluded that  $S$  could be used to compare different experiments for the same nucleus and to choose the most reliable one.

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## 1 Introduction

Information theoretical methods are starting to be important tools for studies of quantum mechanical systems. An example is the application of the Maximum Entropy Principle [1] (MEP) to the calculation of the wave function in a potential [2] using as constraints expectation values of simple observables and reconstructing a quantum wave function from a limited set of expectation values. The idea behind the MEP is to choose the least biased result, compatible with the constraints of the problem. Thus the MEP provides the least biased description consistent with the available relevant information. This is done by employing a suitably defined information entropy (IE) that measures the lack of information associated with the distribution of a quantum state over a given known basis. A measure of the IE is Shannon's information entropy  $S$  [3]. For a continuous probability distribution  $p(x)$  ( $\int p(x)dx = 1$ )  $S$  is defined

$$S = - \int p(x) \ln p(x) dx$$

Shannon's IE has played an important role in the study of quantum mechanical systems, in clarifying fundamental concepts of quantum mechanics and in the synthesis of probability densities in position and momentum space [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. An important step was the discovery of an entropic uncertainty relation [4]. For a three-dimensional system it has the form

$$S = S_r + S_k \geq 3(1 + \ln \pi) \simeq 6.434 \quad (k = p/\hbar), \quad (1)$$

where

$$S_r = - \int \rho(\mathbf{r}) \ln \rho(\mathbf{r}) d\mathbf{r}, \quad S_k = - \int n(\mathbf{k}) \ln n(\mathbf{k}) d\mathbf{k}$$

are Shannon's IEs in coordinate and momentum space and  $\rho(\mathbf{r})$ ,  $n(\mathbf{k})$  are the density distribution (DD) and momentum distribution (MD), respectively, normalized to 1.

Inequality (1) is an information-theoretical uncertainty relation stronger than Heisenberg's [4] and does not depend on the unit of length in measuring  $\rho(\mathbf{r})$  and  $n(\mathbf{k})$ , i.e. the sum  $S = S_r + S_k$  is invariant to uniform scaling of coordinates, while the individual entropies  $S_r$  and  $S_k$  are not. The physical meaning of  $S$  is that it is a measure of quantum-mechanical uncertainty and represents the information content of a probability distribution, in our case of the nuclear density and momentum distributions. Inequality (1) provides a lower bound for  $S$  which is attained for Gaussian wave functions.

We quote March who refers to the information entropy with the following words: "Further work is called for before the importance of  $S_r$  and  $S_k$  in atomic theory can be assessed". [17, 18] We could extend that statement for fermionic and correlated bosonic systems as well.

Shannon's IEs  $S_r$  and  $S_k$  have been recently studied for the densities of various systems [9, 11, 12]: the nucleon DD of nuclei, the valence electron DD of metallic clusters and the DD of correlated Bose alkali atoms. It has been found that the same functional form  $S = a + b \ln A$  for the entropy sum as function of the number of particles  $A$  holds approximately for the above systems in agreement with Ref. [5] for atomic systems. Another interesting result [19] is the fact the entropy of an  $N$ -phonon state subjected to Gaussian noise increases linearly with the logarithm of  $N$ . In Refs. [14, 15] the dependence of  $S$  on the short-range correlations (SRC) parameter of the nucleons in nuclei, and of the particle interaction in various uniform Fermi systems (nuclear matter,  $^3\text{He}$  liquid, and electron gas) has been found. This dependence as well as the linear dependence of  $S$  on  $\ln A$  were used in Ref. [14] to determine the SRC parameter of nucleons in  $s$ -,  $p$ -, and  $sd$ -shell nuclei. In Ref. [10] another definition of IE according to phase-space considerations [20] was used and an information theoretical criterion for the quality of a nuclear DD was derived, i.e. the larger  $S$  the better the quality of the nuclear model. In Ref. [21] the DD, the MD and the Shannon's IEs were calculated for nuclei using three different cluster expansions. The parameters of the various expressions were determined by least-squares fit of the theoretical charge form factors to the experimental ones. It was found that the larger the entropy sum, the smaller is the value of  $\chi^2$ , indicating that the maximal  $S$  is a criterion of the quality of a given nuclear model according to the MEP. Only two exceptions to that rule were found out of many cases examined. Before proceeding, it is appropriate to mention that additional applications of entropy have attracted interest in recent years [22, 23], but in a different spirit, in nuclear physics problems, such as in analysis of shell-model eigenvectors. The authors in Ref. [23] defined a correlational entropy. We note, however, that this is a von Neumann

entropy, which they applied in the framework of the nuclear shell model. In our case we use the definition of the IE according to Shannon applied to the density distribution in coordinate and in momentum space of a nuclear system. Finally, alternative measures of the IE have been proposed by Oniscu [24] and by Brukner and Zeilinger [25].

The motivation of the present work is to extend our previous study of IE in nuclei using as many experimental data available in the literature as possible. In this way, it will be possible to study important features of realistic nuclear systems, e.g. the effects of nucleon-nucleon correlations on basic nuclear characteristics, such as density and momentum distributions. Thus, instead of starting our study from a given nuclear model we start from the "experimental" DDs from Refs. [26] and [27] (obtained from electron scattering by nuclei and muonic atoms) or/and from estimations of "experimental" MDs from Refs. [28, 29] based on superscaling analysis [30] of inclusive electron scattering from nuclei as well as on the coherence density fluctuation model (CDFM) [31, 32, 33, 34] which helps to calculate the MD in connection with the DD and vice versa. This study helps in two ways:

First, it helps to strengthen the empirical property  $S = a + b \ln A$  which has been proposed in previous studies for various fermion systems [5, 9, 11, 12]. Also it enables us to strengthen the conclusion of Ref. [10] that the more experimental data we use in the theory the larger is the IE of a nucleus, i.e. the larger  $S$  the better the quality of the DD of a nucleus is. For the five models which we applied to various nuclei from  ${}^4\text{He}$  to  ${}^{238}\text{U}$  it was found that the same functional form of  $S$  holds, while the values of the parameters  $a$  and  $b$  depend on the model. The harmonic oscillator (HO) shell-model is closer to the lower limit of inequality (1) which is attained only for the  ${}^4\text{He}$  nucleus. The results for more realistic models deviate from the HO ones. The deviation becomes larger when more experimental data are included in the model. The various cases we considered can be classified according to either the quantity of the experimental data we use or the values of  $S$ , i.e., the increase of the quality of the density and of the momentum distributions leads to an increase of the values of  $S$ . As the DD and the MD based on experimental data should be considered as the least biased ones, our results are in accordance with the MEP. Another characteristic result is that the various lines  $S_{\text{model}} = a_{\text{model}} + b_{\text{model}} \ln A$  are almost parallel, i.e., there is a kind of scaling of the values of  $S$  for the various models.

Second, our work helps to connect the IE with other physical quantities such as the root-mean-square (RMS) radius of the nucleus or the mean volume in which the nucleons are confined in a nucleus. If Shannon's IE is considered as ensemble entropy, this connection could be used to relate the mean volume of the nucleus or the mass number with the "ensemble volume" through the relation  $V_{\text{ensemble}} = e^S$  [35]. This ensemble volume provides a direct measure of uncertainty of the system which is advantageous when one wishes to compare the spread of two ensembles of a given type. Finally, the present work might also help to choose the most reliable experiment when more than one experiments are made for the same nucleus.

The paper is organized as follows: In Sec. II we briefly review the theory of the CDFM and describe the way in which the IEs  $S_r$ ,  $S_k$  and their sum can be calculated from the DD or/and the MD. In Sec. III the numerical calculations of IEs for various nuclei, using three different approaches, are presented and compared with previous calculations. The possibility of choosing the most reliable experiment, when more than one experiments are made for the same nucleus, is also discussed. In Sec. IV, a dependence of the IE sum on the mean volume of the nucleus is proposed and a connection between the ensemble

volume of the nucleus and the mean volume or the mass number is given. Finally, Sec. V contains our conclusions.

## 2 The information entropy from a given distribution

In order to find the information entropy of various nuclei, the distributions of the density and the momentum should be known. These distributions can be found using various models. Another way is to employ given DDs, i.e., the phenomenological ones given e.g. from Refs. [26, 27] or the "experimental" MDs using the superscaling analysis in nuclei [30, 28, 29]. The idea is to use the CDFM to find the MD in connection with the DD or to find the DD in relation to the MD of a nucleus.

The CDFM [31, 32, 33, 34] is related to the  $\delta$ -function limit of the generator coordinate method [36] in which the  $A$ -body wave function of a nucleus is written in the form

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_A) = \int F(x_1, x_2, \dots) \Phi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_A; x_1, x_2, \dots) dx_1 dx_2 \dots, \quad (2)$$

where the generating function  $\Phi(\{\mathbf{r}_i\}; x_1, x_2, \dots)$  depends on the coordinates of the nucleons (radius-vector, spin, isospin) and on the generator coordinates  $x_1, x_2, \dots$ .  $\Phi$  is usually chosen to be a Slater determinant built up from single-particle wave functions corresponding to a given construction potential parameterized by the generator coordinates. The weight function  $F(x_1, x_2, \dots)$  can be determined (using the variational principle) as a solution of the Hill-Wheeler integral equation

$$\int [\mathcal{H}(x, x') - E\mathcal{I}(x, x')] F(x') dx' = 0, \quad (3)$$

where  $\mathcal{H}(x, x') = \langle \Phi(\{\mathbf{r}_i\}, x) | \hat{H} | \Phi(\{\mathbf{r}_i\}, x') \rangle$  and  $\mathcal{I}(x, x') = \langle \Phi(\{\mathbf{r}_i\}, x) | \Phi(\{\mathbf{r}_i\}, x') \rangle$ ,  $i = 1, 2, \dots, A$  and  $x$  denotes a set of  $x_1, x_2, \dots$ .

In the CDFM the DD and the MD are expressed by means of the same weight function  $F(x)$

$$\rho(\mathbf{r}) = \int_0^\infty \frac{3A}{4\pi x^3} |F(x)|^2 \Theta(x - |\mathbf{r}|) dx \quad (4)$$

and

$$n(\mathbf{k}) = \frac{4}{(2\pi)^3} \int_0^\infty \frac{4\pi x^3}{3} |F(x)|^2 \Theta(k_F(x) - |\mathbf{k}|) dx \quad (5)$$

both normalized to the mass number  $A$

$$\int \rho(\mathbf{r}) d\mathbf{r} = A, \quad \int n(\mathbf{k}) d\mathbf{k} = A. \quad (6)$$

$\Theta$  is the unit step function,  $x$  is the generator coordinate and  $k_F(x)$  is the Fermi momentum of a piece of nuclear matter with radius  $R = x$

$$k_F(x) = \left( \frac{3\pi^2}{2} \rho_0(x) \right)^{1/3} = \frac{\alpha}{x}, \quad \alpha = \left( \frac{9\pi A}{8} \right)^{1/3}. \quad (7)$$

For DD normalized to  $A$  the weight function obeys the constraint

$$\int_0^\infty |F(x)|^2 dx = 1. \quad (8)$$

Various paths can be followed to find the function  $F(x)$ . Here we follow the approach proposed in Refs. [31, 32, 33] and used also in [28, 29]. As can be seen from Eq. (4), for known DD the weight function  $F(x)$  can be determined by

$$|F(x)|^2 = -\frac{1}{\rho_0(x)} \frac{d\rho}{dr} \Big|_{r=x}, \quad (9)$$

where  $\rho_0(x) = \frac{3A}{4\pi x^3}$  and  $\rho(r)$  satisfies the constraint  $\frac{d\rho}{dr} \leq 0$ .

Substituting  $|F(x)|^2$  from Eq. (9) in the right-hand side of Eq. (5),  $n(k)$  takes the form

$$n(k) = \frac{8}{9\pi A} \left[ 6 \int_0^{\alpha/k} r^5 \rho(r) dr - \left( \frac{\alpha}{k} \right)^6 \rho \left( \frac{\alpha}{k} \right) \right]. \quad (10)$$

Eq. (10) shows the MD as a functional of the density distribution. This point is discussed in Refs. [33, 37] within the framework of the density functional theory.

Thus, within the CDFM the MD of a nucleus can be found from Eq. (10). From  $\rho(r)$ , and  $n(k)$  the information entropies  $S_r$ ,  $S_k$  defined by the relations

$$S_r = -\int \rho(\mathbf{r}) \ln \rho(\mathbf{r}) d\mathbf{r}, \quad S_k = -\int n(\mathbf{k}) \ln n(\mathbf{k}) d\mathbf{k}, \quad (11)$$

and their sum  $S = S_r + S_k$  can be calculated. We note that for the calculation of  $S_r$  and  $S_k$  we use DD and MD normalized to 1.

In the recent paper [29] the momentum distribution of nuclei  $^4\text{He}$ ,  $^{12}\text{C}$ ,  $^{27}\text{Al}$ ,  $^{56}\text{Fe}$ , and  $^{197}\text{Au}$  was calculated using the analysis of the superscaling phenomenon in inclusive electron scattering from nuclei [30]. From those distributions and from Eq. (5) the weight function  $F(x)$  can be calculated as

$$|F(x)|^2 = -\frac{1}{n_0(x)} \frac{dn}{dk} \Big|_{k=x}, \quad (12)$$

where  $n_0(x) = \frac{3A}{4\pi x^3}$  and  $n(k)$  satisfies the constraint  $\frac{dn}{dk} \leq 0$ .

Substituting  $|F(x)|^2$  from Eq. (12) in the right-hand side of Eq. (4), the DD can be expressed by

$$\rho(r) = \frac{8}{9\pi A} \left[ 6 \int_0^{\alpha/r} k^5 n(k) dk - \left( \frac{\alpha}{r} \right)^6 n \left( \frac{\alpha}{r} \right) \right]. \quad (13)$$

Thus, within the CDFM the DD of a nucleus can be estimated approximately by means of the MD. Using  $\rho(r)$  and  $n(k)$  the entropies  $S_r$ ,  $S_k$ , and  $S$  can be calculated from Eqs. (11).

We note the symmetry of the expressions (10) and (13) for both equivalent basic characteristics, the nucleon momentum and local density distributions.

### 3 Numerical results and discussion

For the calculation of the information entropy of a nucleus we used three different approaches. In the first one we used the "experimental" DDs for various nuclei from  $^4\text{He}$

to  $^{238}\text{U}$  from Refs. [26, 27]. For the various DDs existing in the literature we used only two or three parameter Fermi (2pF, 3pF) distributions from [26]

$$\rho(r) = \rho_0 \frac{1 + wr^2/c^2}{1 + \exp[(r - c)/\alpha]} \quad (14)$$

( $w = 0$  in the 2pF distributions), and the symmetrized Fermi distributions from [27]

$$\rho(r) = \rho_0 \frac{\sinh(c/\alpha)}{\cosh(r/\alpha) + \cosh(c/\alpha)}. \quad (15)$$

The reason we avoided the use of other phenomenological distributions is that there usually exist oscillations in the central region of the densities of the nuclei which destroy the constraint  $d\rho/dr \leq 0$ . This is not the case for the Fermi-type distributions.

From those distributions and from Eq. (10) the MD for various nuclei can be found. In the second approach we used the "experimental" MD from the superscaling analysis of Ref. [29] for the nuclei  $^4\text{He}$ ,  $^{12}\text{C}$ ,  $^{27}\text{Al}$ ,  $^{56}\text{Fe}$ , and  $^{197}\text{Au}$ . As shown in [29], in the CDFM the MD is approximately related to the  $\psi'$ -scaling function  $f(\psi')$  (introduced in [30]) by

$$n(k) = -\frac{1}{3\pi k^2 k_F} \left. \frac{\partial f(\psi')}{\partial(|\psi'|)} \right|_{|\psi'|=k/k_F} \quad (16)$$

where  $k_F$  is the Fermi momentum which can be calculated within the CDFM [28, 29]. Using the experimental data for  $f(\psi')$  obtained from inclusive electron scattering from nuclei [30] we estimated the MD  $n(k)$ . From those MDs, using Eq. (13) the DD of these nuclei were found. In the third approach we used the "experimental" DDs, for the above mentioned five nuclei, as in the first approach and for the MDs the "experimental" values from the superscaling analysis as in the second approach.

The evaluated values of  $S = S_r + S_k$  in the three approaches are shown by points in Fig. 1a. One can see that  $S$  obeys the universal property

$$S = S_r + S_k = a + b \ln A. \quad (17)$$

The same holds for the IEs  $S_r$  and  $S_k$  which obey the relation

$$S_{r,k} = a_{r,k} + b_{r,k} \ln A \quad (18)$$

The parameters  $a$ ,  $b$  and  $a_{r,k}$ ,  $b_{r,k}$  were determined by least squares fit of the values of  $S$  and  $S_r$ ,  $S_k$  calculated from Eqs. (17) and (18), respectively, to the corresponding evaluated values of the IEs from Eqs. (11). Their values found in the three approaches are shown in Table I. In Figs. 1b and 1c, where the lines  $S_r(A)$  and  $S_k(A)$  (Eq. (18)) are shown, we have not displayed the calculated values of  $S_r$  and  $S_k$ , from Eqs. (11) as the values of the various points are very close to each other in many cases.

For completeness we also give in Table I the corresponding values of those parameters for the HO case, and for the case of short-range correlated DD and MD which were found with three different expansions of the one-body density matrix in Ref. [21]. We mention that the values of  $S_r$ ,  $S_k$ , and  $S$  were found in [21] for the  $s$ -,  $p$ -, and  $sd$ -shell nuclei for DD and MD normalized to  $A$ . From those values of  $S_r$ ,  $S_k$  and  $S$  using the relation [11]

$$S_{r,k}[norm = 1] = \frac{1}{A} S_{r,k}[norm = A] + \ln A \quad (19)$$

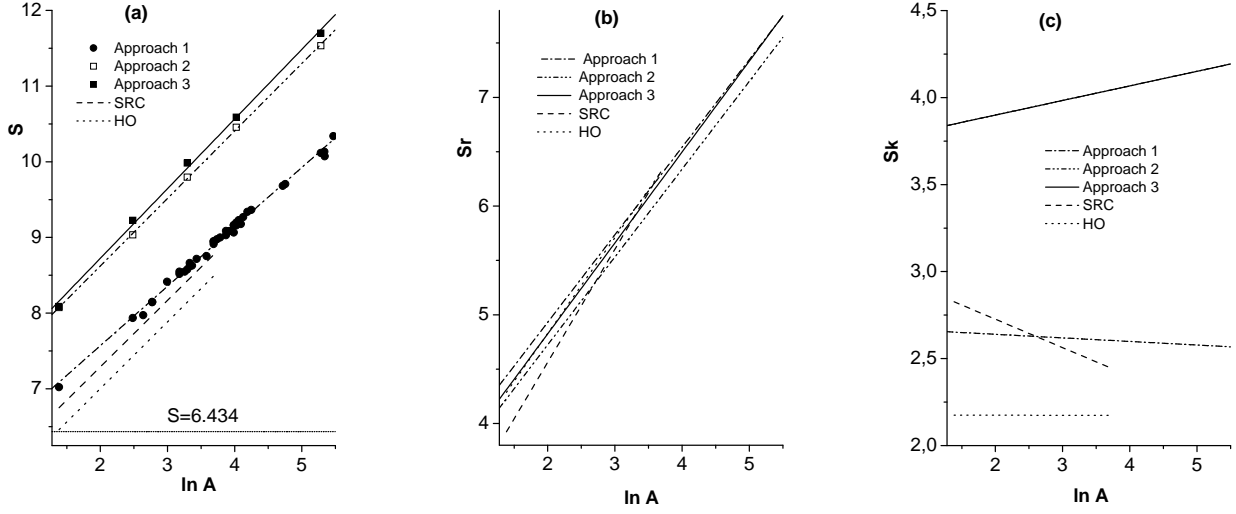


Figure 1: The information entropies in nats: (a)  $S = S_r + S_k$ , (b)  $S_r$ , and (c)  $S_k$  for various nuclei versus the logarithm of the mass number  $A$ . The lines correspond to the fitting expressions  $S = a + b \ln A$  and  $S_{r,k} = a_{r,k} + b_{r,k} \ln A$ , respectively. For the various cases see text. The limiting line corresponding to the lower bound  $S = 6.434$  is also shown.

the corresponding IEs for DD and MD normalized to 1 were found. The values of the IEs  $S_r$ ,  $S_k$  and  $S$  for the three approaches as well as for the HO and the SRC approaches calculated using the values of the parameters  $a$  and  $b$  of Table I are also shown in Fig. 1 by the corresponding lines.

The various cases we examined can be placed in order either by the quantity of the experimental data that were used or by the values of the information entropy obtained for the various nuclei. This is a consequence of Fig. 1a and also of the following discussion.

In the HO model there is only one free parameter which can be determined either by the experimental RMS charge radius or by fit of the theoretical charge form factor ( $F_{ch}(q)$ ) to the experimental one. The results of the HO case presented in Fig. 1 were obtained in Ref. [21] by fit of the theoretical  $F_{ch}(q)$  to the experimental data. In this

Table 1: The values of the parameters  $a$ ,  $b$ ,  $a_{r,k}$ ,  $b_{r,k}$ ,  $a_V$ , and  $b_V$  of relations (17), (18), and (21) in the various cases.

Case	$a$	$b$	$a_r$	$b_r$	$a_k$	$b_k$	$a_V$	$b_V$
HO	5.2391	0.8816	3.0633	0.8822	2.1758	-0.0006	3.4783	0.9472
SRC	5.5330	0.8778	2.4807	1.0409	3.0524	-0.1631	4.2723	0.8512
Approach 1	6.0011	0.7847	3.3205	0.8053	2.6807	-0.0205	4.1166	0.9272
Approach 2	6.8396	0.8919	3.1078	0.8080	3.7318	0.0836	2.0255	1.3969
Approach 3	6.8845	0.9201	3.1527	0.8362	3.7318	0.0839	4.8413	1.0635

case only the low momentum transfer data are reproduced. If we include SRC as in Ref. [21] there are two free parameters which are determined by fit of the theoretical  $F_{ch}(q)$  to the experimental one. In this case more diffraction minima are reproduced than in the HO case. This is reflected in the values of  $S$ . The inequality  $S_{SRC} > S_{HO}$  holds for all the  $s$ -,  $p$ -, and  $sd$ -shell nuclei we have examined.

In the first approach of the present work the Fermi-type distributions, which are employed, are phenomenological distributions reproducing better the electron scattering experiments than in the previous two cases. This is reflected in the values of  $S$ . For all nuclei from  ${}^4\text{He}$  to  ${}^{238}\text{U}$  we have examined, the inequality  $S_{approach1} > S_{SRC}$  holds.

In the second approach the "experimental" data of the MD [29] were used. In this case the high-momentum components of the MD were included in the calculations in a more reliable way than in the first approach. This results in an increased contribution of  $S_k$  to the information entropy sum. Thus, the inequality  $S_{approach2} > S_{approach1}$  holds for the five nuclei we have examined.

Finally, in the third approach information from experimental data both from DD [26, 27] and from MD [29] were included. This leads to increased values of  $S$  in comparison with the corresponding values in the second approach where only information from the "experimental" MD were taken into account in the calculation of  $S$ .

Thus, we should conclude that the increase of the quality of the DD or/and of the MD leads to an increase of the values of the information entropy sum. As the DD and the MD based on experimental data should be considered as the least biased ones, the previous statement is in accordance with the MEP. Another characteristic feature of  $S_{model}$  can be seen from Fig. 1a, i.e. the lines  $S_{model} = a_{model} + b_{model} \ln A$  are almost parallel, i.e. there is a kind of scaling of the values of  $S$  for the various cases we have examined.

From Figs. 1b and 1c and from the values of the parameters  $b_r$  and  $b_k$  it can also be concluded that  $S_r$  depends strongly on  $\ln A$  while  $S_k$  does not. Thus, the linear dependence of  $S$  on  $\ln A$  is mainly due to the IE in coordinate space. The strong dependence of  $S_r$  on  $\ln A$  should be related to the volume of the nucleus where the nucleons are confined. The weak dependence of  $S_k$  on  $\ln A$  is related to the fact that the high-momentum components of the MD are independent of the mass number of the nucleus. This is a well known fact (see, e.g. [33, 28, 29]).

Another feature of Fig. 1b is that  $S_r$  does not depend strongly on the model which is used. The relative difference of the values of  $S_r$  obtained in the various cases is about 10% or less.  $S_k$  (Fig. 1c) depends strongly on the model which is employed. It becomes larger when the model includes high-momentum components of the MD which are related to the presence of SRC. The relative difference of  $S_k$  for the various cases we have examined is about 50%.

From Fig. 1a (see also Fig. 2 in Sec. IV) one can see that the points corresponding to five nuclei in approaches 2 and 3 lie almost on the lines  $S = a + b \ln A$ . This is not the case in approach 1. In this approach while most of the points are on the line  $S = a + b \ln A$ , there are few of them (e.g. the points corresponding to nuclei  ${}^{14}\text{N}$ ,  ${}^{27}\text{Al}$  and  ${}^{209}\text{Bi}$ ) which are relatively far from that line. For these nuclei the distances of the evaluated values of  $S$  from the line  $S = a + b \ln A$  are within the errors of the parameters of the corresponding Fermi distributions. It is also mentioned that for  ${}^{209}\text{Bi}$  the 2pF distribution reproduces only the low momenta transfer of the electron scattering in the  $q$ -range  $= 0.07 - 0.53 \text{ fm}^{-1}$ . [26]

In Ref. [26] there are cases where various 2pF and 3pF distributions reproduce dif-

ferent experimental data for the same nucleus. The deviations of some points from the line  $S = a + b \ln A$  in approach 1 lead us to examine these distributions by comparing the evaluated values of  $S$  for the same nucleus. In Table II we give the calculated values of  $S$  for various nuclei with Fermi distributions from the analysis of different experiments and the corresponding ranges of the momentum transfer. It is seen that in almost all the

Table 2: The values of the information entropy  $S$  for various nuclei in approach 1. The calculations were made with the phenomenological 2pF or/and 3pF distributions of Ref. [26].

Nucleus	$S$ [nats]	$q$ -range [fm <sup>-1</sup> ]	Nucleus	$S$ [nats]	$q$ -range [fm <sup>-1</sup> ]
<sup>19</sup> F	8.3890	0.55-1.01	<sup>64</sup> Zn	9.2948	0.30-1.09
<sup>19</sup> F	8.3947	0.46-1.79	<sup>64</sup> Zn	9.2603	0.15-0.79
<sup>20</sup> Ne	8.3977	0.22-1.04	<sup>66</sup> Zn	9.3376	0.96-1.63
<sup>20</sup> Ne	8.4137	0.21-1.12	<sup>66</sup> Zn	9.2881	0.15-0.79
<sup>20</sup> Ne	8.4262	0.49-1.80	<sup>68</sup> Zn	9.3253	0.96-1.63
<sup>24</sup> Mg	8.5162	0.58-1.99	<sup>68</sup> Zn	9.2920	0.15-0.79
<sup>24</sup> Mg	8.4660	0.74-3.46	<sup>70</sup> Zn	9.3643	0.30-1.09
<sup>24</sup> Mg	8.5175	0.20-1.15	<sup>70</sup> Zn	9.3245	0.15-0.79
<sup>50</sup> Cr	9.0200	0.15-0.79	<sup>142</sup> Nd	9.8978	0.55-2.97
<sup>50</sup> Cr	9.0829	0.97-1.62	<sup>142</sup> Nd	9.8211	0.23-0.59
<sup>52</sup> Cr	9.0243	0.15-0.79	<sup>142</sup> Nd	9.8762	0.22-0.73
<sup>52</sup> Cr	9.0860	0.97-1.62	<sup>146</sup> Nd	9.9897	0.55-2.97
<sup>54</sup> Cr	9.0951	0.15-0.79	<sup>146</sup> Nd	9.8871	0.22-0.73
<sup>54</sup> Cr	9.1623	0.97-1.62	<sup>150</sup> Nd	10.0284	0.55-2.97
<sup>54</sup> Fe	9.0633	0.15-0.79	<sup>150</sup> Nd	9.9305	0.22-0.73
<sup>54</sup> Fe	9.1014	0.97-1.62	<sup>150</sup> Nd	9.8638	0.37-2.29
<sup>56</sup> Fe	9.1113	0.15-0.79	<sup>238</sup> U	10.2478	0.37-0.97
<sup>56</sup> Fe	9.1585	0.97-1.62	<sup>238</sup> U	10.2267	0.46-2.08

cases the larger  $q$ -range corresponds to larger value of  $S$  for the same nucleus. From the many cases of Table II we found only three exceptions which correspond to nuclei <sup>24</sup>Mg, <sup>150</sup>Nd and <sup>238</sup>U. The disagreement of the results for these nuclei to the above rule is due to the following reasons:

In <sup>24</sup>Mg the 3pF distributions which have been used in the cases of momentum transfer ranges:  $q = 0.58 - 1.99 \text{ fm}^{-1}$  and  $q = 0.74 - 3.46 \text{ fm}^{-1}$  give charge distributions which become 0 for relatively small values of the radius ( $r \approx 7 \text{ fm}^{-1}$ ). This is not the case for the 2pF distribution ( $q$ -range=  $0.20 - 1.15 \text{ fm}^{-1}$ ) which becomes 0 for much larger

values of  $r$ . The existence of the logarithm of the density in the integral of  $S_r$  makes the IE sensitive to the tail of the DD. This disagreement could be removed if we used the errors of the parameters of the 3pF distributions.

In the nucleus  $^{150}\text{Nd}$  the value of  $S$  corresponding to the  $q$ -range=  $0.22 - 0.73 \text{ fm}^{-1}$  is larger than the value of  $S$  corresponding to the  $q$ -range=  $0.37 - 2.29 \text{ fm}^{-1}$ . We expected the inverse order. This disagreement should be due to the fact that the real analysis of the electron scattering data was made with a deformed Fermi distribution in the latter case (see the corresponding remark of Ref. [26]). That deformation was not taken into account in our calculations.

In the case of the nucleus  $^{238}\text{U}$  the 2pF distributions do not give the experimental charge RMS radius 5.84 fm and 5.854 fm corresponding to the momentum transfer ranges:  $q = 0.37 - 0.97 \text{ fm}^{-1}$  and  $q = 0.46 - 2.08 \text{ fm}^{-1}$ , respectively. The values, we found, are 5.731 fm and 5.712 fm, respectively. That difference should come from the 2pF distributions we have used, instead of using the deformed Fermi distributions of Refs. [38, 39] (see also the corresponding remarks of Ref. [26]).

Finally, in  $^{142}\text{Nd}$  while the values of  $S$  are increasing with the  $q$ -range, the increase is quite small from  $q$ -range=  $0.22 - 0.73 \text{ fm}^{-1}$  to  $q$ -range=  $0.55 - 2.97 \text{ fm}^{-1}$ . The corresponding values of  $S$  are 9.8762 and 9.8978, respectively. The relatively small values of  $S$  in the latter case is due to the fact that the constraint  $d\rho/dr \leq 0$  does not hold for all the values of  $r$ . This has as a result the high-momentum components of the MD not to be reproduced within the CDFM as correctly as in the former case.

From the above discussion we should conclude that Shannon's information entropy could be used to compare different experiments for the same nucleus and to choose the most reliable one.

## 4 The dependence of $S$ on the mean volume of the nucleus

The strong dependence of  $S_r$  and the nearly independence of  $S_k$  on  $\ln A$  leads us to connect  $S$  with the RMS radius of a nucleus, i.e., with the volume of the nucleus where the nucleons are confined. If we assume spherical symmetry, the mean volume of the nucleus is

$$V = \frac{4\pi}{3} \langle r^2 \rangle^{3/2} = \frac{4\pi}{3} \left[ 4\pi \int_0^\infty r^4 \rho(r) dr \right]^{3/2}. \quad (20)$$

The calculated values of  $S$  versus  $\ln V$  for the various cases we have examined are shown in Fig. 2. It is seen that the information entropy sum depends linearly on the logarithm of  $V$

$$S = a_V + b_V \ln V. \quad (21)$$

The values of the parameters  $a_V$  and  $b_V$  determined by least square fit are given in Table I.

The almost parallel displacement of the lines corresponding to the approaches 1 and 3 is due to the fact that we used the same DDs from Refs. [26, 27] in both approaches. The different slope of the line of the second approach comes from the CDFM we have used to calculate the DDs of the five nuclei from the MD [29].

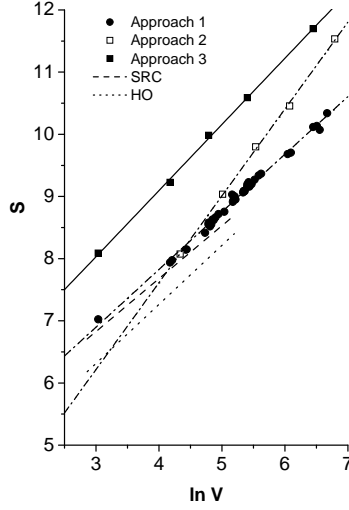


Figure 2: The information entropies in nats for various nuclei versus the mean volume of the nucleus. The lines correspond to the fitting expression  $S = a_V + b_V \ln V$ . For the various cases see text.

In Ref [35] it has been shown that for any ensemble (classical or quantum, discrete or continuous) there is essentially only one measure of the "volume" occupied by the ensemble, which is compatible with basic geometrical notions. This volume is called "ensemble volume" and provides a universal choice or a direct measure of uncertainty, which is advantageous when one wishes to compare the spreads of two ensembles of a given type. The ensemble volume turns out to be proportional to the exponent of the entropy of the ensemble [35], i.e.

$$V_{ensemble} = K(\Gamma) e^S. \quad (22)$$

The constant  $K(\Gamma)$  is a normalization constant reflecting the notion that only relative volumes are of real interest in comparing different ensembles.

Using Eq. (22), the Gibbs relation  $S_{therm} = kS$ , between thermodynamical entropy and ensemble entropy for equilibrium ensembles, can be written as

$$S_{therm} = k \ln[V_{ensemble}/K(\Gamma)], \quad (23)$$

i.e., within an additive constant, the thermodynamical entropy is proportional to the logarithm of the ensemble volume [35]. We can make the same statement between the IE of a nucleus and its mean volume, via Eq. (21).

Assuming that Shannon's IE  $S$  is the ensemble entropy of a nucleus and substituting  $S$  from Eq. (17) into Eq. (22), for  $K(\Gamma) = 1$ , we have

$$V_{ensemble} = cA^b, \quad c = e^a. \quad (24)$$

Thus, the ensemble volume of a nucleus is analogous to  $A^b$ .

It was mentioned in the introduction that Eq. (17) is an universal property of the fermion systems (electrons in the atoms or in metallic clusters, nucleons in nuclei) as well as of correlated bosons in a trap [5, 9, 11, 12]. Thus, Eq. (24) is also valid for atoms, metallic clusters and for correlated bosonic systems.

From the linear dependence of  $S$  on the logarithm of the mean volume of the nucleons, Eq. (21), and from Eq. (22) a relation between the ensemble volume of the nucleus and the mean volume of it can be found. That relation is

$$V_{ensemble} = c_V V^{b_V} , \quad c_V = e^{a_V}. \quad (25)$$

Thus, the ensemble volume of a nucleus is analogous to  $V^{b_V}$ .

From the three approaches we have considered, the first and the third are the most reliable ones because the DD's are based on experimental data. In these two approaches the values of the parameter  $b_V$  are quite close to 1. That is why we could say that the ensemble volume of a nucleus is analogous to the mean volume of the nucleus.

## 5 Conclusions

A study of Shannon's IEs in coordinate space,  $S_r$ , in momentum space,  $S_k$ , and their sum,  $S$ , was made for many nuclei using three approaches based on "experimental" DDs or/and on "experimental" MDs.

In the first approach we used Fermi-type phenomenological DDs reproducing the electron scattering experiments for many nuclei from  $^4\text{He}$  to  $^{238}\text{U}$  [26, 27]. The MDs of these nuclei were found within the CDFM [31, 32, 33, 34]. In the second approach we used the "experimental" MDs from superscaling analysis for the nuclei  $^4\text{He}$ ,  $^{12}\text{C}$ ,  $^{27}\text{Al}$ ,  $^{56}\text{Fe}$  and  $^{197}\text{Au}$ , [29] while the DDs of these nuclei were found within the CDFM. In the third approach we used the "experimental" DDs of the five nuclei as in the first approach and the "experimental" MDs as in the second approach. The DDs and the MDs were used for the evaluation of the IEs  $S_r$ ,  $S_k$  and  $S$ . It was found that in the three approaches  $S_r$ ,  $S_k$  and  $S$  depend linearly on the logarithm of the mass number in accordance with previous studies of various Fermi systems.

The values of  $S$  found in the three approaches of the present work were compared with the ones evaluated with the HO case and the short-range correlated DDs and MDs. It was found that for all the nuclei we have considered the following inequalities hold

$$S_{HO} < S_{SRC} < S_{approach1} < S_{approach2} < S_{approach3}$$

Thus, the various cases can be classified according to either the quantity of the experimental data we used or the values of the IE sum obtained for the various nuclei. In other words the increase of the quality of the DD or/and of the MD leads to an increase of the values of the IE sum according the maximum entropy principle.

It is also found that  $S_r$  depends strongly on  $\ln A$  and does not depend very much on the model we use. The behaviour of  $S_k$  is opposite. The properties of  $S_r$  and  $S_k$  lead us to find that, within an additive constant the information entropy  $S$  is proportional to the logarithm of the mean volume of the nucleus in accordance with the fact that for equilibrium ensembles the thermodynamical entropy is proportional to the ensemble volume[35]. In the case that Shannon's entropy may be considered as ensemble entropy, a connection of the ensemble volume with the mass number of the nucleus or with the mean volume of the nucleus can be found.

Finally, the comparison of the values of  $S$  for the same nucleus using phenomenological DDs from the analysis of different experiments could be used to choose the most reliable one.

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